Decision Making Using Rating Systems: When Scale Meets Binary

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Abstract

Rating systems measuring quality of products and services (i.e., the state of the world) are widely used to solve the asymmetric information problem in markets. Decision makers typically make *binary* decisions such as buy/hold/sell based on aggregated individuals' opinions presented in the form of ratings. Problems arise, however, when different rating metrics and aggregation procedures translate the same underlying popular opinion to different conclusions about the true state of the world. This paper investigates the inconsistency problem by examining the mathematical structure of the metrics and their relationship to the aggregation rules. It is shown that at the individual level, the only *scale* metric (1,...,N) that reports people's opinion equivalently in the a *binary* metric (-1, 0, 1) is one where N is odd and N-1 is not divisible by 4. At aggregation level, however, the inconsistencies persist regardless of which scale metric is used. In addition, this paper provides simple tools to determine whether the binary and scale rating systems report the same information at individual level, as well as when the systems differ at the aggregation level. [1] [2]

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1 Introduction

Rating systems are widely used in business, political systems, and in our daily lives as methods of containing and communicating information that is crucial to the decision making process. For instance, investors make decisions by consulting ratings of financial products, online shoppers compare products by examining seller and product ratings, doctors make diagnoses based on a patient's rating of their well-being, students refer to university rankings to decide which school to attend, and universities use rating systems to monitor professors' performance in teaching. Rating systems provide a way to summarize public opinion in an organized manner. A rating system is defined by a rating metric and an aggregation rule that combines individual's ratings into a single overall score. Several forms of rating systems exist. Two popular systems are the *binary* system and the *scale* system. For example, eBay's reputation system asks users to rate a buyer or seller in a binary rating system $\{-1, 0, 1\}$, where -1 is considered a negative rating, 0 is a neutral rating, and 1 is a positive rating. On the other hand, Amazon.com asks raters to use a scale rating system of $\{1, 2, 3, 4, 5\}$ where one is considered worst and five is best [3]. Using information about the aggregate of individuals' opinions in either the scale or the binary system, a decision maker is required to draw a conclusion about a product or a service. Typically, this conclusion has a binary outcome such as buying, holding, or selling of a stock. Problems and paradoxes arise, however, when the scale rating system leads a decision maker to make one decision but the binary rating system leads the decision maker to a contradictory decision.

To illustrate how the aggregated information from the scale ratings can be contradictory to that of the binary ratings, consider two groups of ten financial analysts rating a stock in both a scale system and a binary system. Using a scale of one to five (strong buy = 5, buy = 4, hold = 3, under perform = 2, and sell = 1), half the analysts in the first group rate the stock as a 4 and the other half rate it as a 2. In a binary system (buy = 1, hold = 0, and sell = -1), the group's collective decision is to "buy" the stock. The second group of ten analysts all rate the stock 3 in the scale system and aggregate to "hold" in the binary system. The scale ratings of both groups suggest the

[3] Throughout this paper, we adopt the terminology used in Bhattacharjee and Goel (2005) to name the different rating systems. They refer to eBay-like systems as "binary feedback systems," while Amazon-like systems are called "scale feedback systems."

overall opinion of the stock is neutral – group one is split 50-50 around 3 and group two all rates 3. On the other hand, in the binary system, the two groups give contradictory collective decisions – group one says "buy" while group two says "hold."

It is hypothesized these types of inconsistencies are mainly due to human errors, such as a person not understanding the meaning of the scale, or a person's strategic dishonest rating. In this paper, we illustrate that even without human errors and strategic ratings, inconsistent decision outcomes can occur. In fact, the two metrics and aggregation procedures can translate the same underlying popular opinion to different decision outcomes due to the restrictions placed by the mathematical structures of the systems.

This paper investigates the inconsistency problem by examining the mathematical structure of the metrics and their relationship to the aggregation rules. We show that at the individual rating level, the only scale metric $\{1, \ldots, N\}$ that can report a person's opinion equivalently as the binary metric $\{-1, 0, 1\}$ is one where N is odd and $N - 1$ is not divisible by 4. At the aggregate level, however, the inconsistencies persist regardless of which scale metric is used. The differences in the aggregation systems are characterized and simple tools are illustrated to determine whether the binary and scale rating systems are reporting the same information both at the individual rater's level as well as at the collective aggregate level.

2 Related Literature

Ranking systems and rating systems are two closely related methods for capturing and summarizing public opinion proposed in the literature. In general, a ranking system asks participants to rank-order alternatives while a rating system asks participants to score an alternative on an arbitrary scale. Research on these systems has focused on obtaining best methods for capturing and representing people's opinions. Connections and distinctions between the two types of systems have also been made in the literature. In both systems, the alternatives can represent the object of opinion or the opinion itself. In the case where the object being summarized is the opinion itself, ranking systems are a special case of rating systems (Droba, 1931). For example, a person may be asked to organize alternatives into three different groups representing the worst, middle, and best. In the case where there are numeric values associated with each category (e.g, worst = 0, middle = 1, and best = 2), each participant ranks according to a rating. These types of ranking methods not only provide relative comparisons among alternatives but they also provide rating information about the alternatives.

Although rating systems are more general than ranking systems, previous work has focused primarily on rankings. The problem of aggregating ranking preferences is one of the most vexing and difficult in economics, political science, and decision science. Issues of existence of desirable aggregation functions date back at least as far as Arrow's Impossibility Theorem (Arrow, 1963). Since then, numerous papers have discussed inconsistencies with voting and social choice rules used for aggregating preferences as well as new interpretations and resolutions (Moulin, 1988, Sen, 1970, 1986, Saari, 2001a, 2001b, Li and Saari, 2008).

As with ranking systems, similar issues and inconsistencies arise when using rating systems to capture and aggregate information as shown in the above example. Two possible explanations have been offered as the cause of the inconsistencies. They are: (1) people misunderstand or misinterpret the rating scale, and (2) people rate strategically.

In marketing, the debate on which scale to use began several decades ago (Lehmann and Hulbert, 1972, McDaniel and Gates, 2006). In recent years, researchers find that people do not differentiate greatly among the scale values and assign high ratings to most items (McCarty and Shrum, 2000, Greenleaf, Bickart, and Yorkston, 1999). Because the rating data does not capture details about people's opinions, it makes it difficult to use in choosing an effective marketing strategy. It is hypothesized that this phenomena occurs because people may not understand the meaning of the rating scale. In social science and psychology research, Alwin and Krosnick (1985) also observe that rating methods do not differentiate well among people's opinion because respondents tend to rate everything high. Landy and Farr (1980) argue that the cognitive characteristics of raters seem to explain the bias toward high ratings in rating systems. For example, changing a scale from -3 to 3 creates different results than a scale from 1 to 7 highlighting the fact that the actual numeric values used in the scale influence respondents' interpretation (Schwarz et al., 1991).

Another possible explanation offered in the literature for the inconsistencies surrounding data is strategic behavior. Cleveland and Murphy (1992) and Murphy et al. (2004) suggest that the reason for raters giving ratings that appear psychometrically suspect is due to their different individual goals when completing performance appraisals. For example, raters may want to maintain harmony within the workgroup or possibly motivate subordinates to perform better in the future. Similar hypotheses have been explored in the economic literature focusing on strategic behavior in information aggregation (see Carwford and Sobel, 1982, Sobel, 2006, Morgan and Stocken, 2008).

Because rating systems are a part of market reputation systems, feedback mediators are an important factor in designing and improving reputation systems (Dellarocas et al., 2006). Relying on a user feedback system that yields poor representations of user's opinions can lead to mistrust and misuse of a particular market. For example, the design of eBay's bi-lateral reputation system leads to problems of bias towards positive ratings (See Dellarocas and Woods, 2006, Klein et al., 2006, and Li forthcoming). In this context, understanding the mathematical structure of the rating system can help determine a successful design for feedback ratings in a market system.

By reviewing the current literature, it can be seen that the existing explanations of the inconsistencies and issues with rating systems are due to human limitations and behaviors. The focus of this current paper is to characterize the inconsistencies that can occur among the binary and scale rating system in terms of the mathematical structure of the systems. While previous contributions explain the "meaning" of the scales and how people interpret them, this paper strives to understand the structural differences among the scales. Even without human errors and strategic behavior, the results show inconsistencies between the binary system and scale rating system persist.

3 Framework

Suppose a decision maker needs to draw a conclusion about a product, service, or trade based on available information about its quality. The information available is in the form of a collection of scale and binary ratings capturing the public's opinion about the product or service. Scale information may offer more details about the intensity of people's opinion while binary information may capture a more direct up-or-down-like measure. Either could prove useful to the decision maker. Using these two types of ratings, the decision maker can make a decisive conclusion. Typically, this conclusion will have a binary outcome such as the decision maker deciding to buy, hold, or sell the product; or the decision maker deciding to use, maybe use, or not use a service. Problems arise, however, when the information conveyed through the scale ratings is not consistent with the information conveyed through the binary rating; therefore leading the decision maker to different conclusions depending on which rating data he/she examines.

The scale and binary rating data collected is a gathering of people's opinions about the product, service, or trade. For example, a committee of faculty members may be asked to give scale and binary ratings representing their opinion of the quality of a particular job candidate. In such instances, the product, in this example, the job candidate, being evaluated possesses an actual true unobservable quality. This true state of the product will be denoted as TS throughout the paper. Although there exists a TS for the product, each rater may perceive its quality differently. For each rater, we define their perception by PS, the perceived state of the product, to represent their individual opinion about TS.

When presented with a rating system, a rater must translate their PS into a rating in the given rating metric. This rating is then what is available to the decision maker to aid in the decision making process. When the information conveyed through the scale system is not consistent with the information conveyed through the binary system, the decision maker may have evidence to support different conclusions depending on which rating data he/she consults.

In order to explore this inconsistency issue, we decompose the rating procedure into two sequential levels – the individual level and the aggregation level. At the individual level, individuals choose a rating score in the available metric closest to their perception PS. At the aggregation level, the individuals' ratings are combined using an averaging function. At the first level, it is reasonable to expect that individuals who express the same opinion in the scale system should express the same opinion in the binary system. In other words, if the individuals' ratings agree in the scale system then they should also agree in the binary system. At the second level, the aggregate results from the scale and binary metrics should lead the decision maker to the same conclusions about TS. These two desirable properties can be summarized as: (1) the representation of individual's opinions in the scale and binary metric must be equivalent (i.e., individuals who express the same opinion in the scale system should express the same opinion in the binary system), and (2) the aggregation of individuals' opinions in each system must be consistent (i.e., lead the decision maker to the same conclusion.) The goal of this paper is to explain these properties mathematically and provide tools for decision makers to identify when the properties are satisfied. To do this, we begin by formally defining a rating system.

A rating system is composed of two separate items: a metric and an aggregation rule. Individuals rate in the metric and then aggregate their scores according to a specified rule. The metric is typically a discrete set of integers contained in an interval. For example, the scale metric $\{1, 2, 3, ..., 10\}$ can be expressed as the set of integers in the interval $[1, 10]$. An aggregation rule combines people's opinions into one overall score. Typically, the rule takes the form of an averaging function. Mathematically, a rating system can be defined in the following manner.

Definition 1 *Let $int[m, n]^k$ be the k-product of the set of integer values in the interval [m,n]. A rating system is a set { int[m,n], R } where int[m,n] is the set of integers between m and n containing m and n. R is a function from $int[m, n]^k$ to the interval [m,n] where k is the number of people rating.*

$$R\text{: } int[m, n]^k \rightarrow [m, n]$$

The binary rating system can therefore be specified as { int[-1,1], R_b } where int[-1,1] = {-1, 0, 1 } and R_b: $int[-1, 1]^k \rightarrow [-1, 1]$. The scale system can be written as { int[1,N], R_s } where int[1,N] = { 1, 2, 3, ..., N } and R_s: $int[1, N]^k \rightarrow [1, N]$.

4 Translating Individual Rating from Scale to Binary

As described above the true state, TS, can be thought of as the true quality of a product or service that exists outside of a decision maker's or rater's opinion. The true state TS is modeled as a point in the interval $[1, N]$.[4] Each rater may perceive TS differently and rate the closest available integer to their perceived state, PS, in the given metric. If PS is halfway between two rating options, then without loss of generality, the rater will choose the higher integer option. [5]

Example 1 *Suppose two people rate a service using the metric* $int[1, 5]$*. Person* A *perceives the service as* 2.1 *and person* B *perceives it at* 1.9*. Since the available metric requires the individuals to rate* $1, 2, 3, 4,$ *or* 5*, both individuals rate* 2*. Both* 2.1 *and* 1.9 *perceived states are closest to the integer* 2*.*

If the same individuals rate in the binary metric, how are their opinions represented? Rating in the binary metric requires an individual to translate their PS from the interval $[1, N]$ to the interval $[-1, 1]$. The perceived state in the interval $[1, N]$ can be translated to a point in the interval $[-1, 1]$ using a linear transformation. Once this step is complete the individuals will again rate at the closest available option to their PS in the binary metric.

Definition 2 *Given metrics* $int[a, b]$ *and* $int[c, d]$*. Any point* $k \in [a, b]$ *can be transformed to a point* $j \in [c, d]$ *by the following linear transformation:*

$$k = \tfrac{da-cb}{d-c} + \tfrac{b-a}{d-c} j \qquad (1)$$

This means that a perceived state PS_1 *in metric* 1 *can be transformed into a perceived state* PS_2 *in metric* 2 *by:*

[4] Typically, people think in a positive continuous interval thus it is natural to capture quality and perception in the positive interval $[1, N]$

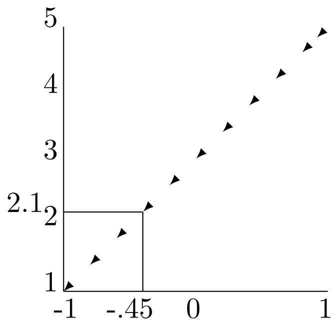
[5] For simplicity and without loss of generality, it is assumed that if an individual has perceived state half way between two integer values, then the individual will cast their rating as the highest value. For example, in a $[1, 10]$ scale metric, if a rater has $PS = 8.5$, then the individual will rate $r = 9$. If a rater has a perceived state of $\frac{1}{2}$ in the binary metric, then the rater will rate 1. This assumption is made to model the findings in the prior literature about ratings being skewed high.

$$PS_1 = \frac{da-cb}{d-c} + \frac{b-a}{d-c} PS_2 \qquad (2)$$

Therefore a perceived state, PS, in metric $[1, N]$ can be transformed to a perceived state in $[-1, 1]$ by $PS_{[1,N]} = \frac{1+N}{2} + \frac{N-1}{2} PS_{[-1,1]}$.

Example 2 *Person A and person B have perceived states, 2.1 and 1.9 respectively. Then, person A has perceived state equal to $PS_{[-1,1]} = (2.1 - \frac{1+5}{2})(\frac{2}{5-1}) = -.45$ in the binary metric. This transformation is modeled in Figure 1. Similarly, person B has perceived state equal to $-.55$. Person A and B may rate -1, 0, or 1 in this metric. Since $-.45$ is closest to 0, A rates 0. Since $-.55$ is closest to -1, B rates -1.*

Figure 1: Transforming scores between metrics



Examples 1 and 2 illustrate how the two metrics can represent the same opinions differently. In the scale metric, person A and B rate the same, however, in the binary metric they rate differently. In this situation, a decision maker examining both the scale and binary ratings is given conflicting information. Ideally, the two metrics would represent rater's opinions in the same manner. We define this desired property of equivalency in the following manner:

Definition 3 *A scale metric and a binary metric are called* **equivalent** *if and only if for every two raters who rate the same in scale metric implies that they also rate the same in binary metric.*

4.1 Scale and Binary Equivalency

Because two metrics can represent the same underlying opinions differently, a natural question to ask is whether there are specific conditions under which the scale and binary will be equivalent (i.e., rater's opinions that are the same in the scale will also be the same in the binary). In order for a discrepancies between the metric's representation of people's opinions to occur, two ratings that round to the same integer in $[1, N]$ must round to different integer values in $[-1, 1]$ after applying the linear transformation. Thus, in order to ensure equivalency, the rounding points in the binary metric must be the image of the rounding points in the scale metric under the linear transformation. The rounding points represent the cut-off points where a rater will either round up or round down. In the binary metric, the rounding points are $-.5$ and $.5$ [6]. Raters having a $PS_{[-1,1]}$ less than $-.5$ will rate -1, those between $-.5$ and $.5$ will rate 0, and those with $PS_{[-1,1]}$ larger than $.5$ will rate 1. In order for the scale and binary metrics to be equivalent, the pre-image of $-.5$ and $.5$ under the linear transformation, given by $\frac{1+\frac{1+N}{2}}{2}$ and $\frac{N+\frac{1+N}{2}}{2}$, must not be not integer values. If they are integer values, then raters will be "split" at the rounding point leaving them to rate the same in the scale metric but not the same in the binary metric. The following two examples highlight this point.

Example 3 *Consider the scale metric* $[1, 5]$. *Under the linear transformation, the pre-image of rounding point* $-.5$ *is the number* 2. *In order to create a discrepancy between the metric's representation of the two opinions, choose two raters who rate differently in the binary metric and are split by rounding point in* $-.5$. *Suppose rater 1 has* $PS_{[-1,1]}$ *equal to* $-.4$ *and rater 2 has* $PS_{[-1,1]}$ *equal to* $-.7$. *In the binary metric, these raters will not rate the same with rater 1 rating 0 and rater 2 rating -1. When applying the inverse of the linear transformation to their opinions, we see that rater 1 and rater 2 agree in* $[1, 5]$ *and both rate* 2. *By Definition* 3, *the* $[1, 5]$ *metric and the binary metric are therefore not equivalent.*

[6] Throughout this paper, we use cut-off points $\frac{-1}{2}$ and $\frac{1}{2}$ for simplicity, however, these can be generalized to any two points $\frac{-1}{s}$ and $\frac{1}{s}$ in the interval $[-1, 1]$. To determine the cut-off points of a particular population of raters, evidence must be collected through empirical work. Because the purpose of this paper is to develop theory, we focus on the midpoint as the cut-off point.

Example 4 *Consider the scale metric* $[1, 7]$. *Under the linear transformation, the pre-image of rounding point* $-.5$ *is* 2.5. *Because the pre-image is not integer valued, raters who are split by a rounding point in the binary will also be split in the scale thus causing them not to rate the same in* $[1, 7]$. *Rater 1 and rater 2 will still rate 0 and -1 respectively in the binary metric but in* $[1, 7]$, *their opinions* $-.4$ *and* $-.7$ *fall on different sides of* 2.5 *leading rater 1 to rate* 2 *and rater 2 to rate* 3. *In this case, the raters rate differently in both in the scale and binary metric.*

In order to ensure equivalency between the two systems, two ratings that round to the same integer in $[1, N]$ *must* round to the same integer values in $[-1, 1]$ after applying the linear transformation. The following theorem summarizes the only case for which this happens. The proof is deferred to the Appendix.

Theorem 1 *The scale system and binary system are equivalent if and only if* N *is odd and* $N - 1$ *is not divisible by* 4.

For all other N, it is possible to pick points in specific intervals that cause the scale and binary metric to be nonequivalent. As an illustration, consider the scale metric with N odd but $N - 1$ not divisible by 4. In this metric, the pre-image under the linear transformation of the cut-off points of $-.5$ and $.5$ are the integer values $\frac{1+\frac{1+N}{2}}{2}$ and $\frac{N+\frac{1+N}{2}}{2}$ respectively. Choose raters 1 and 2 such that their PS's lie on either side of $\frac{1+\frac{1+N}{2}}{2}$ but within the interval $[\frac{1+\frac{1+N}{2}}{2} - \frac{1}{2}, \frac{1+\frac{1+N}{2}}{2} + \frac{1}{2}]$, then the raters will both rate at the integer value $\frac{1+\frac{1+N}{2}}{2}$ in the scale metric but will be split in the binary metric because their PS's will fall on either side of the $-.5$ cut-off point. Similarly, when N is even, one can always choose rater perceived states in such a way that they round to the same integer in the scale metric but are split in the binary.

Therefore, Theorem 1 completely characterizes all inconsistencies among raters at the individual level. The characterization points out how the metrics represent information differently and can cause discrepancies to occur.

4.2 Implications of Differences between Binary and Scale Metric at the Individual Rater Level

Except for the case where N is odd and is not divisible by 4, the binary and the scale metric interpret ratings in a different manner. What are the implications of the differences between the systems for these cases? Do the systems tend to penalize or inflate the ratings? In Example 1 and 2 above, person A's PS is represented as a 0 rating in the binary system and a 2 in the scale system. Because the 2 rating is below the neutral rating in the [1,5] scale (i.e., below the midpoint of 3) and 0 is the neutral rating in the binary metric, the binary system actually represents A's PS at a higher value than the scaled. The opinion in the binary system is thus interpreted as higher than A's opinion in the scale. This means that with perceived state equal to 2.1 in $[1, 5]$, the binary system actually inflates the score to 0 in $[-1, 1]$. To the contrary, person B's perceived state (1.9 in $[1, 5]$) is penalized in the binary system. Person B rates -1 in the binary system that has a lower interpretation than the 2 in the scale. The following theorem follows directly from Definition 2 to characterize the regions of the interval [1, N] that the binary system distorts.

Theorem 2 *For a fixed N, let PS_s equal a rater's perceived state in the scale system and r_s be its equivalent rating given by r_s=$\{k$: $k\in$ [1,N] and min$|PS_s$-$k|\}$. Let x_{sb} equal the translation of r_s into the binary system. Note that x_{sb} may not be a rating and thus not be an integer. Let PS_b be the translation of PS_s in the binary system and r_b be it's rating given by r_b=$\{j$: $j\in$ [-1,1] and min$|PS_b$-$k|\}$. The penalty or benefit associated with every rating is given by r_b-x_{sb} if using the binary system instead of the scale system.*

In order to compute that amount of distortion created by the metric, the difference in what the rating should be (i.e., the translated ranking x_{sb}) and the actual binary rating r_b) is computed. The following example illustrates the computation.

Example 5 *Suppose $N = 5$. Let $PS_s \in [1.5, 2]$ be given. Then the individual will rate r_s= 2 in the int[1,5] metric. The 2 rating in the scale system is equivalent to the position at x_{sb}=-0.5 in* *the binary system. However, in the binary system, an individual with PS* $\in [1.5, 2]$ *will rate* r_b*=-1. Therefore,* r_b-$x_{sb}$$= -0.5$ *penalty in the binary system.*

If PS $\in [2, 2.5]$*, the individual will rate as a* r_s*= 2 in the scale system. As before, the rating 2 is equivalent to the position* x_{sb}=-0.5 *in the binary system. However, for PS* $\in [2, 2.5]$*, the individual will rate* r_b*=0. Therefore,* r_b-x_{sb}*= 0.5 benefit in the binary system.*

Theorem 3 *The only scale system for which the binary system* **does not distort** *people's opinion is the case where N is odd and N-1 is not divisible by 4.*

The theorem follows as a direct consequence of Theorems 1 and 2. This underscores the importance of scale selection in applications.

5 Aggregation Procedures Using Different Metrics

Once individuals have cast their ratings, the information is aggregated to form an overall score. An aggregation rule, R, is typically defined as an average or weighted average function. Two issues can arise when aggregating ratings: (1) the average score in the scale system may not be consistent with the average score computed in the binary system, and (2) the decision the average score in the scale system may lead one to make is not the same as the decision the average score in the binary system leads one to make. These two issues may arise even when using a scale metric where N is odd and $N-1$ is not divisible by 4. Consider the following example to motivate the aggregation discussion.

Example 6 *Mike is interviewed for a job and* 10 *people in the office are asked to rate him on a scale of* 1-7*. Among the* 10 *people,* 4 *have perceived states near* 4 *and* 6 *have perceived states around* 6*. In addition to rating Mike on a scale, each of the* 10 *people is asked directly whether they would like to hire Mike, whether they would like to wait and see the next candidate, or whether they would not like to hire Mike (corresponding to ratings* 1*,* 0*, and* -1 *respectively). In the* 1-7 *scale* *system, the aggregated average score* $= 5.2$ *(computed as (4(4) + 6(6)) /10). This score translates to a score of* $.4$ *in the binary system which yields the decision to wait and see the next candidate before making a decision about Mike. In the binary system,* 4 *people choose to wait and see the next candidate while the other* 6 *raters choose to hire him. The aggregated average score in this system is* $.6$ *(computed as (0(4) + 1(6)) /10). According to the binary aggregation, an offer is extended to Mike.*

The example illustrates that even when using a scale where N is odd and $N-1$ is not divisible by 4, the aggregated ratings may not be consistent. In addition, the decisions the aggregated scores lead to may also be different. The following sections provide simple tools to check when these inconsistencies occur.

5.1 Issue 1: Average Scores Are Not Consistent

In the binary system, let b_1, b_2, and b_3 represent the number of $-1, 0, 1$ ratings respectively. In the scale system, let s_1, s_2,..., s_N represent the number of $1, 2, ..., N$ ratings respectively. The average function R_b aggregates the binary ratings as: $\frac{b_3-b_1}{b_1+b_2+b_3}$ and R_s aggregates the scale ratings as: $\frac{\sum_{i=1}^{N} is_i}{\sum_{i=1}^{N} s_i}$. Thus, R_b is defined as a function from $int[-1,1]^k \to [-1,1]$ and R_s is a function from $int[1,N]^k \to [1,N]$. Comparing the two average functions amounts to understanding the values b_1, b_2, b_3 and s_1, s_2,..., s_N must equal in order for the two rules to output consistent scores.

Definition 4 *Let R_x: $int[a,b]^k \to [a,b]$ and R_y: $int[c,d]^k \to [c,d]$. For each of the k raters, denote r_{xi} and r_{yi} as rater $i's$ ratings in $int[a,b]$ and $int[c,d]$ respectively. Then, R_x and R_y are called* **consistent** *for the group of k raters if and only if the aggregated score in one metric is equal to a linear transformation of the aggregated score in the other metric. i.e., R_x and R_y satisfy the following equation:*

$$R_y(r_{y1}, \ldots, r_{yk}) = \frac{bc-da}{b-a} + \frac{d-c}{b-a} R_x(r_{x1}, \ldots, r_{xk}). \tag{1}$$

For R_x and R_y being the average function, then the equation can be written as:

$$\frac{1}{k}\sum_{i=1}^{k} r_{yi} = \frac{bc-da}{b-a} + \frac{d-c}{b-a}\left(\frac{1}{k}\sum_{i=1}^{k} r_{xi}\right). \tag{2}$$

Because different aggregation procedures have different domains and ranges, they are only considered consistent if their output is the same under the linear transformation. In the case of the binary and scale systems, the average functions are consistent provided the aggregated binary score is equal to the linear translation of the aggregated scale score. This means that rules R_b: $int[-1,1]^k \rightarrow [-1,1]$ and R_s: $int[1,N]^k \rightarrow [1,N]$ must satisfy the following equation:

$$\frac{\sum_{i=1}^{N} i s_i}{\sum_{i=1}^{N} s_i} = \frac{1+N}{2} + \frac{N-1}{2}\left(\frac{b_3-b_1}{b_1+b_2+b_3}\right) \tag{3}$$

for a set of individuals rating in both the binary and scale metrics. Here are two examples to illustrate that the consistency of R_b and R_s depend on the perceived states and ratings given by the k raters.

Example 7 *Suppose two people rate the same event in both a* 1-5 *scale system and binary system* $-1, 0, 1$*. The following table lists their perceived states and their respective scores in each metric.*

Raters	*Perceived State in [1,5]*	*Scale Metric Vote*	*Binary Metric Vote*
Rater 1	*1.1*	*1*	*-1*
Rater 2	*2.7*	*3*	*0*

*The aggregation rule in the binary system, computes an overall score of R_b(-1, 0)= $\frac{-1+0}{2} = \frac{-1}{2}$ and the rule in the scale system computes R_s(1, 3)=$\frac{1(1)+3(1)}{2} = 2$. By Eq.(**??**), these aggregation results are consistent.*

Example 8 *Again, the following table lists the perceived states of the two raters. They differ slightly from the previous example only in the first rater's perceived state.*

Raters	*Perceived State in [1,5]*	*Scale Metric Vote*	*Binary Metric Vote*
Rater 1	*1.6*	*2*	*-1*
Rater 2	*2.7*	*3*	*0*

The binary rule tallies R_b*(-1,0)=* $\frac{-1+0}{2} = \frac{-1}{2}$ *and the scaled rule tallies* R_s*(2,3)=* $\frac{2(1)+3(1)}{2} = 2.5$*. These aggregation results are not consistent by Eq. (**??**).*

These examples show how consistency between R_b and R_s is dependent on the ratings, which, in turn, are dependent on the number of rater's perceived states in certain regions of [-1,1] and [1,N].

Definition 5 *Given the interval [1,N] where N is odd, define region* $A=(\frac{1+(\frac{N+1}{2})}{2}-\frac{1}{2}, \frac{1+(\frac{N+1}{2})}{2})$*, region* $B=(\frac{1+(\frac{N+1}{2})}{2}, \frac{1+(\frac{N+1}{2})}{2}+\frac{1}{2})$*, region* $C=(\frac{N+(\frac{N+1}{2})}{2}-\frac{1}{2}, \frac{N+(\frac{N+1}{2})}{2})$*, and region* $D=(\frac{N+(\frac{N+1}{2})}{2}, \frac{N+(\frac{N+1}{2})}{2}+\frac{1}{2})$*. Let* $|X|$ *denote the number of voter's perceived states in region X.*

Definition 6 *Given the interval [1,N] where N is even and divisible by 4, define region* $A=(\frac{1+\frac{1+N}{2}}{2}-\frac{1}{4}, \frac{1+(\frac{N+1}{2})}{2})$*, region* $B=(\frac{1+(\frac{N+1}{2})}{2}, \frac{1+\frac{1+N}{2}}{2}+\frac{3}{4})$*, region* $C=(\frac{\frac{1+N}{2}+N}{2}-\frac{3}{4}, \frac{N+(\frac{N+1}{2})}{2})$*, and region* $D=(\frac{N+(\frac{N+1}{2})}{2}, \frac{N+(\frac{N+1}{2})}{2}+\frac{1}{4})$*. If N is greater than 2 and not divisible by 4, define region* $A=(\frac{1+\frac{1+N}{2}}{2}-\frac{3}{4}, \frac{1+(\frac{N+1}{2})}{2})$*, region* $B=(\frac{1+(\frac{N+1}{2})}{2}, \frac{1+\frac{1+N}{2}}{2}+\frac{1}{4})$*, region* $C=(\frac{\frac{1+N}{2}+N}{2}-\frac{1}{4}, \frac{N+(\frac{N+1}{2})}{2})$*, and region* $D=(\frac{N+(\frac{N+1}{2})}{2}, \frac{N+(\frac{N+1}{2})}{2}+\frac{3}{4})$*. Let* $|X|$ *denote the number of voter's perceived states in region X.*

The following theorems characterize the differences in aggregation procedures by stating conditions the rater's perceived states must meet in order to ensure consistency between R_b and R_s. The proofs are deferred to the Appendix.

Theorem 4 *For N odd, let regions A, B, C, and D be defined as in Definition 5. If* $|A| + |C| = |B|+|D|$*, then* R_b *and* R_s *will be consistent.*

The left side of the equation, |A| + |C|, counts the number of rating that are rounded up in the scale system but are rounded down in the binary system. That is, a perceived state in region A will be rounded up to the closest integer to $\frac{1+\frac{1+N}{2}}{2}$ rating in the scale but when transformed into a binary rating, it will be rounded down to -1. In region C, the scale rating will round up to the closest integer to $\frac{N+\frac{1+N}{2}}{2}$ but the binary rating will round down to 0. The right hand side of the equation, |B|+|D|, counts the opposite – the number of ratings that are rounded down in the scale and rounded up in the binary. If the two sides of the equation are equal, the average will be the same in both scales.

Theorem 5 *For N even, let regions A, B, C, and D be defined as in Definition 6. If* $|A|=|D|$ *and* $|B|=|C|$*, then* R_b *and* R_s *will be consistent.*

Theorem 5 follows a similar argument as Theorem 4. It exemplifies how for the even case a "reverse" type symmetry in the perceived states of the raters ensures consistency between R_b and R_s. If |A|=|D| and |B|=|C|, then the scale ratings will aggregate to the midpoint $\frac{1+N}{2}$. Since the raters in region B and C rate 0 while the voters in region A rate -1 and those in region D rate 1, the symmetric balance of the scale ratings is carried over to the binary system. If there are an equal number of raters who have perceived states at -1 and 1, then the average aggregated score is 0. Scoring 0 in the binary system is exactly equivalent to scoring the midpoint $\frac{1+N}{2}$ in the scale system. By combining Theorems 4 and 5 and because perceived states falling in regions A, B, C, and D cause problems, we obtain the following general result.

Theorem 6 *If all voter's perceived state are in [1, N] - (A$\cup$ B $\cup$ C $\cup$ D) then* R_b *and* R_s *will be consistent.*

These results provide conditions that rater's perceived states must satisfy in order to ensure binary and scale systems are consistent. These conditions exploit the difference in metrics of the two systems. Because in practice there is no restriction on people's beliefs, people can have a perceived state anywhere in the interval [1,N], however, we illustrate that depending on the location of the perceived states, the systems do not necessarily utilize the rating in the same way, only in very special cases do they do so.

5.2 Issue 2: Decisions Are Not Consistent

The consistency condition introduced in Definition 4 above is quite strict. In fact, it is possible for the same decision to be reached when using scale ratings and binary ratings even when the aggregation functions do not output consistent scores. The following provides an example.

Example 9 *Alice, Iris, and Mike all have purchased a book from a seller online. Using a* 1-7 *scale, Alice perceives the transaction as* 2.4*, Iris perceives* 2.8*, and Mike perceives* 5.3*. They rate* 2*,* 3*, and* 5 *respectively. By linearly translating their perception, they rate* -1*,* 0*, and* 0 *in the binary system. Because the aggregated score in the scale system is* 3.33 *(computed as (*2+3+5*)/3)) and the aggregated score in the binary system is* $-.33$ *(computed as (*-1+0+0*)/3), by Definition* 4 *the scores are not consistent (i.e., the linear transformation of* 3.33 *is* $-.22$ *not* $-.33$*). However, because both* $-.22$ *and* $-.33$ *are closest to* 0*, both scores yield a neutral decision about the seller.*

The consistency condition guarantees that the systems will lead a decision maker to the make the same decision when consulting both scale and binary information. However, because we really only interested in ensuring a "consistent" decision rather than a consistent score, the condition can be relaxed. The following theorem characterizes the conditions that the scale and binary ratings must meet in order to obtain consistent decisions.

Theorem 7 *For k raters, Let r_{si} represents rater i's rating in the* 1-N *scale system and r_{bi} represents rater i's rating in* $-1, 0, 1$ *binary system. Given the rounding points $\frac{-1}{2}$ and $\frac{1}{2}$ in the binary system, the aggregation will lead to different decision if the ratings in the binary and scale system satisfy any of the following four conditions:*

$$\text{(1) } \sum_{i=1}^{k} r_{si} \geq \frac{\frac{-k}{2}(N-1)+k(N+1)}{2} \text{ and } \sum_{i=1}^{k} r_{bi} < \frac{-k}{2}$$

$$(2)\ \sum_{i=1}^{k} r_{si} < \frac{\frac{-k}{2}(N-1)+k(N+1)}{2} \ \ and \ \sum_{i=1}^{k} r_{bi} \geq \frac{-k}{2}$$
$$(3)\ \sum_{i=1}^{k} r_{si} \geq \frac{\frac{k}{2}(N-1)+k(N+1)}{2} \ \ and \ \sum_{i=1}^{k} r_{bi} < \frac{k}{2}$$
$$(4)\ \sum_{i=1}^{k} r_{si} < \frac{\frac{k}{2}(N-1)+k(N+1)}{2} \ \ and \ \sum_{i=1}^{k} r_{bi} \geq \frac{k}{2}$$

The inconsistent decisions occur when the aggregated opinions are on different sides of the cut-off points -0.5 and 0.5 in binary system. Cases (1) and (2) describe the scale and binary ratings that lead to discrepancies around -0.5 while cases (3) and (4) describe those around 0.5. This results provides a way to determine whether a different decision will be reached when consulting the binary or scale ratings.

6 Conclusion

Rating systems measuring quality of products or services are typically used to solve the asymmetric information problem in markets. They summarize public opinion and are widely used to stabilize markets, ensure quality of service, and help people assess situations. A rating system is composed by a metric and an aggregation rule. The results in this paper illustrate how different metrics and aggregation procedures may translate the same opinion to different conclusions. Previous literature has explained these differences as a result of human error or purposeful intention. However, these results show that even without human error or strategically dishonest ratings, the inconsistencies among systems still exist.

This work characterizes the differences in the binary and scale systems by highlighting how the two metrics may represent an individual's opinion differently and how aggregating ratings in the two metrics may lead to inconsistent decision outcomes. In particular, at the individual level, we showed that the binary system is equivalent to the scale system only in the case where N is odd and $N-1$ is divisible by 4. At the aggregation level, simple tools were found that determine whether the binary and scale rating systems are reporting the same information. We found that when rater perceptions are located in certain regions of $[1, N]$, then the aggregation results in both systems lead to the same outcome.

The results presented provide a new and simple mathematical explanation for the inconsistencies found among rating systems as well as provide new tools to study preference information aggregation in social choice and reputation system design. This paper is a first attempt to understand rating systems from a mathematical point of view. Future work and direction can include incorporating strategic rating behavior and selection of a rating system that minimizes the difference between the aggregated rating score and the true state.

7 Proofs

Theorem 1:

If N is odd and N-1 is not divisible by 4, then $\frac{1+\frac{1+N}{2}}{2}$ is not an integer. By definition 2, $\frac{1+\frac{1+N}{2}}{2}$ transformed into the binary metric is -0.5. If two rater rate smaller than $\frac{1+\frac{1+N}{2}}{2}$, then both will rate -1 in the binary. If two raters rate larger than $\frac{1+\frac{1+N}{2}}{2}$, then both raters will rate 0 in the binary metric. Because $\frac{1+\frac{1+N}{2}}{2}$ is not an integer and has the form k.5 where k is an integer, then the closest integers are $\frac{1+\frac{1+N}{2}}{2} - \frac{1}{2}$ and $\frac{1+\frac{1+N}{2}}{2} + \frac{1}{2}$. If two raters rate $\frac{1+\frac{1+N}{2}}{2} - \frac{1}{2}$, then their perceived states will be in $[\frac{1+\frac{1+N}{2}}{2} - 1, \frac{1+\frac{1+N}{2}}{2}]$. Any two raters with perceived states in this interval will rate -1 in binary. If two raters rate $\frac{1+\frac{1+N}{2}}{2} + \frac{1}{2}$, then their perceived states will be in $[\frac{1+\frac{1+N}{2}}{2}, \frac{1+\frac{1+N}{2}}{2} + 1]$. Any two raters with perceived states in this interval will rate 0 in binary. The $\frac{N+\frac{1+N}{2}}{2}$ follows in the exact same manner. Therefore, the systems are equivalent everywhere.

Theorem 4:

Consider the case where N is odd and N-1 is not divisible by 4. For this case, $\frac{1+\frac{1+N}{2}}{2}$ and $\frac{N+\frac{1+N}{2}}{2}$ will be integers and raters with perceived states in region A and B will rate $\frac{1+\frac{1+N}{2}}{2}$ while raters in regions C and D will rate $\frac{N+\frac{1+N}{2}}{2}$. In the binary system, the raters in region A will rate -1, raters in regions B and C will rate 0, and raters in region D will rate 1. The aggregated ratings in the binary systems will thus equal $\frac{-a+d}{a+b+c+d}$ where a = |A|, b = |B|, c = |C|, d = |D|. The aggregated ratings in the scale system will simplify to $\frac{3a+Na+3cN+c+3b+bN+3dN+d}{4(a+b+c+d)}$. Because |A| + |C| = |B| + |D|, then by substituting $a = b + d - c$ the quantity simplifies even further to $\frac{2d+3b-c+2Nd+bN+cN}{2(a+b+c+d)}$. Applying the linear transformation equation (3) to the binary aggregated rating, one obtains $\frac{(1+N)(a+b+c+d)+(N-1)(-a+d)}{2(a+b+c+d)}$. This simplifies to exactly the aggregated scale results equal to $\frac{2d+3b-c+2Nd+bN+cN}{2(a+b+c+d)}$. Thus, if |A| + |C| = |B| + |D|, then the systems are consistent.

The N odd and N-1 divisible by 4 can be proven computationally in the exact same way. In the case of N odd and N-1 divisible by 4, $\frac{1+\frac{1+N}{2}}{2}$ and $\frac{N+\frac{1+N}{2}}{2}$ will not be integers. Thus, raters with perceived states in each region will rate at the closest available integers, namely $\frac{1+\frac{1+N}{2}}{2}$ - $\frac{1}{2}$ for region A, $\frac{1+\frac{1+N}{2}}{2}$ + $\frac{1}{2}$ for region B, $\frac{N+\frac{1+N}{2}}{2}$ - $\frac{1}{2}$ for region C, and $\frac{N+\frac{1+N}{2}}{2}$ + $\frac{1}{2}$ for region D. The computations directly follow in the same way as above.

Theorem 5:

Suppose N is even and divisible by 4 and |A|=|D|=k where k is an integer. Then the k raters in A will rate $\frac{1+\frac{1+N}{2}}{2}$ since this is the closest integer available while the k raters in D will rate $\frac{N+\frac{1+N}{2}}{2}$. Then the aggregation function R_s=$\frac{k(\frac{1+\frac{1+N}{2}}{2})+k(\frac{N+\frac{1+N}{2}}{2})}{2k}$ = $\frac{1+N}{2}$. Using Definition 2, this value is equal to 0 in the binary metric. Now, the k raters with perceived states in region A will rate -1 in the binary metric while the k raters with perceived states in region D will rate 1. This means that R_b= $\frac{k(-1)+k(1)}{2k}$= 0. Therefore, the aggregation functions are consistent. Similarly for regions B and C and when N is greater than 2 and not divisible by 4.

Theorem 7:

Inconsistent decisions occur when the aggregated opinions in the scale and binary systems are on different sides of the cut points -0.5 and 0.5 in the binary system. To create an inconsistent decision, the aggregate of the scale ratings must fall on the opposite side of the cut point than the aggregate of the binary ratings. Conditions (1) and (2) demonstrate this for the lower cut point of -0.5 and conditions (3) and (4) demonstrate for the upper cut point of 0.5.

Consider the lower cut point of -0.5. If the aggregated scale ratings fall on the left side of -0.5 and the binary fall on the right side, then different decisions will be reached. In order for the aggregated scale ratings to be on the left side of -0.5, then $\frac{1}{k}\sum_{i=1}^{k} r_{si} \geq \frac{1+\frac{1+N}{2}}{2}$ and in order for the aggregated binary ratings to be on the right side, then $\frac{1}{k}\sum_{i=1}^{k} r_{bi} < \frac{-1}{2}$. Rewriting $\frac{1+\frac{1+N}{2}}{2}$ = $\frac{\frac{-1}{2}(N-1)+(N+1)}{2}$ and multiplying through by the number of raters k, this is exactly condition (1). The other three cases follow in the same manner.

References

[1] Alwin, F. D. and Krosnick, A. J., (1985). The Measurement of Values in Surveys: A Comparison of Ratings and Rankings, *The Public Opinion Quarterly*, **49** (4), 535-552.

[2] Arrow, K. J., (1963). *Social Choice and Individual Values*, John Wiley Sons, Inc., New York, London, Sydney.

[3] Cleveland, J. N. and Murphy, K. R., (1992). Analyzing performance appraisal as goal-directed behavior. In G. Ferris K. Rowland (Eds.), *Research in personnel and human resources management* (Vol. 10, pp. 121185). Greenwich, CT: JAI Press.

[4] Crawford, V.P. and Sobel, J., (1982). Strategic Information Transmission. *Econometrica*, **50**, 1431-1451.

[5] Dellarocas, C., Dini, F., and Spagnolo, G., (2006). Designing reputation (feedback) mechanisms. In *Handbook of Procurement*, Cambridge University Press, 2006.

[6] Dellarocas, C. and Wood, C. A. The sound of silence in online feedback: Estimating trading risks in the presence of reporting bias. *Management Science*, Forthcoming.

[7] Droba, D. D., (1931). Methods Used for Measuring Public Opinion, *The American Journal of Sociology*, **73** (3), 410-423.

[8] Greenleaf, E A., Bickart, B., and Yorkston, E. A., (1999). How Response Styles Weaken Correlations from Rating Scale Surveys, Working paper, Stern School of Business, New York

[9] Klein, T. J. , Lambertz, C., Spagnolo, G., and Stahl, K. O., (2006). Last minute feedback. CEPR Discussion Paper No. 5693. (2006)

[10] Landy, F. J. and Farr, J. L., (1980). Performance Rating, *Psychological Bulletin*, **87** (1), 72-107.

[11] Lehmann, D. and Hulbert, J., (1972). Are Three-Point Scales Always Good Enough?, *Journal of Marketing Research* **9**, (4), 444-446.

[12] Li, L. I., "Reputation, Trust, and Rebates: How Online Auction Markets Can Improve Their Feedback Systems," *Journal of Economics and Management Strategy*, (forthcoming) .

[13] Li, L. I. and Saari, D. G., (2008). Sens theorem: geometric proof, new interpretations, *Social Choice and Welfare*, 31 (2008), 393-413.

[14] McCarty, J. and Shrum, L.J., (2000). The Measurement of Personal Values in Survey Research: A Test of Alternative Rating Procedures, *The Public Opinion Quarterly*, **64**, (3), 271-298.

[15] McDaniel, C. and Gates, R., (2004). *Marketing Research*, Wiley; 6 edition

[16] Morgan, J. and Stocken, P., (2008). Information Aggregation in Polls. *American Economic Review*, **98**, (3), 864-896.

[17] Murphy, K. R., Cleveland, J. N., Skattebo, A. L., and Kinney, T. B., (2004). Raters Who Pursue Different Goals Give Different Ratings, *Journal of Applied Psychology*, **89** (1), 158-164.

[18] Saari, D.G., (1999). Explaining all three alternative voting outcomes, *Journal of Economic Theory*, **87**, 313-355

[19] Saari, D.G., (2001a) *Chaotic elections! A mathematician looks at voting Providence*, American Mathematical Society.

[20] Saari, D. G., (2001b). *Decisions and Elections; Explaining the Unexpected*, Cambridge University Press, New York.

[21] Schwarz, N., Knauper, B., Hippler, H., Noelle-Neumann, E., and Clark, L., (1991). Rating Scales: Numeric Values May Change the Meaning of Scale Labels, *The Public Opinion Quarterly*, **55** (4), 570-582.

[22] Sen, A. K., (1970a). *Collective Choice and Social Welfare*, Holden-Day, San Francisco.

[23] Sen, A. K., (1970b). The Impossibility of a Paretian Liberal, *The journal of Political Economy*, **78**(1), 152-57.

[24] Sen, A. K., (1986) Social Choice Theory; in *Handbook of Mathematical Economics, Vol. III;* Ed. K.J. Arrow and M. Intriligator, Amsterdam: North-Holland.

[25] Sobel, J., (2006) Information Aggregation and Group Decisions, Working Paper, University of California, San Diego.

[26] Stevens, S. S., (1946). On the Theory of Scales of Measurement, *Science*, **103** (2684), 677-680